A Finite Volume Solver for Radiation Hydrodynamics in the Non Equilibrium Diffusion Limit

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Abstract We derive an Implicit Explicit finite volume scheme for the computation of radiation hydrodynamics. The convective part is handled through a classical upwind method while the reactive and diffusive parts are discretized thanks to a centered scheme. These results are compared to semi-analytic solutions obtained by Lowrie and Edwards [10].

Key words: Cell centered finite volume scheme, Radiation hydrodynamics, Implicit/Explicit schemes
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1 Introduction

Radiation hydrodynamics models are of interest for many applications *e.g.* astrophysics, inertial confinement fusion (ICF) and other flows with very high temperatures. One of the major difficulties for these multi-physics problems is the presence of multiple time scales. From the numerical point of view, this leads to build implicit-explicit schemes with respect to time. The implicit part is here to handle small time scales while the explicit one takes care of larger time scales. In our context, the small time scales result from the radiation transport part (diffusion) while larger time scales come from purely hydrodynamical phenomena (entropy and pressure waves). Our strategy consists in relying on classical cell centered Finite Vol-

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ume schemes based on approximate Riemann solver (namely Flux Schemes see Ghidaglia [5]) for the hydrodynamics part and on an implicit Finite Volume scheme for the radiative one.
This article is a first step towards the derivation of a multi-material solver, that is studying flows with two or more different materials. For example in the ICF applications, we have at least two materials in presence, a metal (Gold) and a highly compressed gas (a mixture of Deuterium and Tritium). The multi material version of the scheme (Chauveheid [3]), relies on a generalization of the method of Braeunig *et al.* [1]. The latter method computes sharp interfaces between non miscible materials whose computation uses directional splitting. Hence in this paper, although we solely address the case of one material, we shall use cartesian meshes.
The governing equations, in non dimensional form (Lowrie and Edwards [10], Lowrie and Morel [9]), read in $3D$ as:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0\,, \tag{1}$$

$$\frac{\partial (\rho \mathbf{u})}{\partial t} + \nabla \cdot \left(\rho \mathbf{u} \otimes \mathbf{u} + \left(p + \mathscr{P}_0 \frac{\mathscr{E}_r}{3} \right) Id \right) = 0\,, \tag{2}$$

$$\frac{\partial (\rho E)}{\partial t} + \nabla \cdot ((\rho E + p)\mathbf{u}) = -\mathscr{P}_0 \left(\sigma (T^4 - \mathscr{E}_r) + \mathbf{u} \cdot \nabla \frac{\mathscr{E}_r}{3} \right), \tag{3}$$

$$\frac{\partial \mathscr{E}_r}{\partial t} + \nabla \cdot (\mathscr{E}_r \mathbf{u}) + \frac{\mathscr{E}_r}{3} \nabla \cdot \mathbf{u} = \nabla \cdot (\kappa \nabla \mathscr{E}_r) + \sigma (T^4 - \mathscr{E}_r)\,, \tag{4}$$

where we denote by ρ the density, $\mathbf{u}$ the velocity field, p the hydrodynamic pressure, related to the density ρ and the internal energy e by an equation of state : $EOS(p, \rho, e) = 0$. The hydrodynamic specific energy $E = e + \frac{1}{2}\|\mathbf{u}\|^2$ is the sum of the specific internal energy e and the kinetic energy, T is the material temperature. The radiative energy is denoted by $\mathscr{E}_r$ and we define the radiation temperature by $T_r^4 = \mathscr{E}_r$. Finally, $\mathscr{P}_0$ is a non dimensional number ([10], [9]). This system is non conservative but adding (3) and $\mathscr{P}_0$ (4) we readily obtain the total energy conservation law:

$$\frac{\partial (\rho E + \mathscr{P}_0 \mathscr{E}_r)}{\partial t} + \nabla \cdot \left(\left(\rho E + p + 4 \mathscr{P}_0 \frac{\mathscr{E}_r}{3} \right) \mathbf{u} \right) = \mathscr{P}_0 \nabla \cdot (\kappa \nabla \mathscr{E}_r)\,. \tag{5}$$

Then, introducing the radiative entropy (as done in [2]) $S_r \equiv T_r^3$, we can rewrite (4) as

$$\frac{\partial S_r}{\partial t} + \nabla \cdot (S_r \mathbf{u}) = \frac{3}{4T_r} \left[\nabla \cdot (\kappa \nabla \mathscr{E}_r) + \sigma (T^4 - \mathscr{E}_r) \right]. \tag{6}$$

The system (1), (2), (5) and (6) is conservative as far as convection terms are concerned. Equation (6) is a non linear heat equation for the radiative temperature T_r. This variable is therefore diffused and the non conservative product appearing in the right hand side of this equation should not induce non uniqueness of solutions.

2 Numerical scheme

We use an operator splitting which consists in solving first the left-hand side of (1), (2), (5) and (6) by means of an upwind explicit finite volume scheme. Then, the diffusion-reaction part is discretized thanks to a centered implicit finite volume scheme. This kind of technique is often referred to as IMEX method (for Implicit/Explicit), see for example [7], [8].
We consider a regular cartesian grid and split also the space differential operators, that is to say we solve successively the x-derivative terms, the y-derivative terms and the z-derivative term.
Therefore, and without loss of generality, we deal only with $1D$ schemes, corresponding to what is done direction by direction. From now on, we call x the generic direction that we are looking at.

2.1 Cell centered upwind Finite Volume scheme for the convection operator

We denote by $v=(\rho,\rho\mathbf{u},\rho E+\mathscr{P}_0\mathscr{E}_r,S_r)$ the conservative variables for the convective part of the system (1), (2), (5) and (6), and $F(v)$ the flux matrix such that:

$$F(v)\cdot\mathbf{n}\equiv(\rho(\mathbf{u}\cdot\mathbf{n}),\rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n})+(p+\mathscr{P}_0\mathscr{E}_r/3)\,\mathbf{n},S_r(\mathbf{u}\cdot\mathbf{n}))\,, \tag{7}$$

is the normal flux in the direction $\mathbf{n}\in\mathbb{S}^{d-1}$, d being the physical space dimension. With these notations, the left-hand side of equations (1)-(2)-(5)-(6) reads:

$$\frac{\partial v}{\partial t}+\nabla\cdot F(v)=0\,. \tag{8}$$

The integration of (8) over a control volume $K_{i,j,k}=[x_i,x_{i+1}]\times[y_j,y_{j+1}]\times[z_k,z_{k+1}]$, keeping only the terms corresponding to the derivation in the generic x-direction, leads to a system of ordinary differential equations:

$$\frac{dV_{K_{i,j,k}}}{dt}+\frac{1}{|K_{i,j,k}|}\left(A_{i+1/2,j,k}\phi(v_{i+1,j,k},v_{i,j,k})-A_{i-1/2,j,k}\phi(v_{i,j,k},v_{i-1,j,k})\right)=0\,, \tag{9}$$

where $\phi(v_{i+1,j,k},v_{i,j,k})$ denotes the numerical flux at the interface between volumes $K_{i,j,k}$ and $K_{i+1,j,k}$. $A_{i+1/2,j,k}$ is the measure of the edge located at $x_{i+1/2}\equiv\frac{x_i+x_{i+1}}{2}$.
The Characteristic Flux Finite Volume (CFFV) scheme. The CFFV scheme [4] consists in choosing, for the numerical flux in (9), the following value:

$$\phi(v,w,\mathbf{n})=\frac{F(v)+F(w)}{2}\cdot\mathbf{n}-\mathscr{U}(u,v,\mathbf{n})\frac{F(w)-F(v)}{2}\cdot\mathbf{n}\,. \tag{10}$$

Here, $\mathbf{n} = e_x$, for the generic x-direction. $\mathscr{U}(u,v,\mathbf{n})$ is the sign matrix of the jacobian $\frac{\partial F(v)\cdot\mathbf{n}}{\partial v}$, in the sense that it has the same eigenvectors as, and its eigenvalues are the signs of those of $\frac{\partial F(v)\cdot\mathbf{n}}{\partial v}$. Namely, when $\frac{\partial F(v)\cdot\mathbf{n}}{\partial v}$ reads $L(diag(\lambda_i))R$ (which is the case for hyperbolic problems), with λ_i the eigenvalues, R right eigenvectors, and L left eigenvectors such that $LR = Id$, we have $\mathscr{U}(u,v,\mathbf{n}) = L(diag(sign(\lambda_i)))R$.
The boundary conditions use the normal flux method, we refer to [6].
Eigenelements. The jacobian matrix of the normal flux (7) is found to be equal to:

$$\frac{\partial F(v)\cdot\mathbf{n}}{\partial v} = \begin{pmatrix} 0 & \mathbf{n} & 0 & 0 \\ K\mathbf{n} - \mathbf{u}(\mathbf{u}\cdot\mathbf{n}) & \mathbf{u}\otimes\mathbf{n} - k\mathbf{n}\otimes\mathbf{u} + (\mathbf{u}\cdot\mathbf{n})Id & k\mathbf{n} & \frac{4}{9}\mathscr{P}_0 T_r(1-3k)\mathbf{n} \\ (K-(H+\frac{4\mathscr{P}_0\mathscr{E}_r}{3\rho}))\mathbf{u}\cdot\mathbf{n} & (H+\frac{4\mathscr{P}_0\mathscr{E}_r}{3\rho})\mathbf{n} - k(\mathbf{u}\cdot\mathbf{n})\mathbf{u} & \mathbf{u}\cdot\mathbf{n}(k+1) & \frac{4}{9}\mathscr{P}_0 T_r(1-3k)\mathbf{u}\cdot\mathbf{n} \\ -\frac{T_r^3}{\rho}\mathbf{u}\cdot\mathbf{n} & \frac{T_r^3}{\rho}\mathbf{n} & 0 & \mathbf{u}\cdot\mathbf{n} \end{pmatrix}.$$

Its eigenvalues are as follows:

$$\begin{cases} \lambda_1(v,\mathbf{n}) = \mathbf{u}\cdot\mathbf{n} - c_s\,, \\ \lambda_2(v,\mathbf{n}) = \cdots = \lambda_{d+2}(v,\mathbf{n}) = \mathbf{u}\cdot\mathbf{n}, \\ \lambda_{d+3}(v,\mathbf{n}) = \mathbf{u}\cdot\mathbf{n} + c_s\,. \end{cases} \tag{11}$$

with $k = \frac{1}{\rho T}\left(\frac{\partial p}{\partial s}\right)_\rho$, $c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$, s being the material entropy, $H = E + \frac{p}{\rho}$, $K = c^2 + k(\|\mathbf{u}\|^2 - H)$ and $c_s^2 = c^2 + \mathscr{P}_0\frac{4\mathscr{E}_r}{9\rho}$.
The right eigenvectors associated to these eigenvalues can be taken equal to:

$$\begin{cases} r_1(v,\mathbf{n}) = (1, \mathbf{u} - c_s\mathbf{n}, H + \frac{4\mathscr{P}_0\mathscr{E}_r}{3\rho} - c_s\mathbf{u}\cdot\mathbf{n}, \frac{T_r^3}{\rho})\,, \\ r_{d+1}(v,\mathbf{n}) = (1, \mathbf{u}, H - \frac{c^2}{k}, 0)\,, \\ r_{d+2}(v,\mathbf{n}) = (\mathscr{P}_0 T_r, \mathscr{P}_0 T_r\mathbf{u}, \mathscr{P}_0 T_r(H - 3c^2), -\frac{9}{4}c^2)\,, \\ r_{d+3}(v,\mathbf{n}) = (1, \mathbf{u} + c_s\mathbf{n}, H + \frac{4\mathscr{P}_0\mathscr{E}_r}{3\rho} + c_s\mathbf{u}\cdot\mathbf{n}, \frac{T_r^3}{\rho})\,, \\ r_2(v,\mathbf{n}) = (0, \mathbf{n}_2^\perp, \mathbf{u}\cdot\mathbf{n}_2^\perp), \cdots, r_d(v,\mathbf{n}) = (0, \mathbf{n}_d^\perp, \mathbf{u}\cdot\mathbf{n}_d^\perp)\,. \end{cases} \tag{12}$$

where $\mathbf{n}_2^\perp \cdots \mathbf{n}_d^\perp$ is an orthonormal basis of the hyperplane orthogonal to $\mathbf{n}$.
The dual basis is then:

$$\begin{cases} \ell_1(v,\mathbf{n}) = \frac{1}{2c_s^2}(K + c_s\mathbf{u}\cdot\mathbf{n}, -k\mathbf{u} - c_s\mathbf{n}, k, \frac{4}{9}\mathscr{P}_0 T_r(1-3k))\,, \\ \ell_{d+1}(v,\mathbf{n}) = \frac{k}{c^2}(H - \|\mathbf{u}\|^2, \mathbf{u}, -1, \frac{4}{9}\mathscr{P}_0 T_r))\,, \\ \ell_{d+2}(v,\mathbf{n}) = \frac{4}{9\rho c^2 c_s^2}(T_r^3 K, -kT_r^3\mathbf{u}, kT_r^3, -\rho c^2 - \frac{4}{9}\mathscr{P}_0 k\mathscr{E}_r)\,, \\ \ell_{d+3}(v,\mathbf{n}) = \frac{1}{2c_s^2}(K - c_s\mathbf{u}\cdot\mathbf{n}, -k\mathbf{u} + c_s\mathbf{n}, k, \frac{4}{9}\mathscr{P}_0 T_r(1-3k))\,, \\ \ell_2(v,\mathbf{n}) = (-\mathbf{u}\cdot\mathbf{n}_2^\perp, \mathbf{n}_2^\perp, 0, 0), \cdots, \ell_d(v,\mathbf{n}) = (-\mathbf{u}\cdot\mathbf{n}_d^\perp, \mathbf{n}_2^\perp, 0, 0)\,. \end{cases} \tag{13}$$

Time discretization and stability condition. We use the explicit Euler's scheme to discretize the time derivative in (9) and then the Courant condition for the linearized scheme reads:

$$\max_{i,j,k}|\lambda^n_{i,j,k}|\frac{A_{i,j,k}}{|K_{i,j,k}|}\Delta t^n \leqslant CFL \leqslant 1, \tag{14}$$

where $A_{i,j,k}$ is either $\Delta x_i \Delta y_j$, $\Delta y_j \Delta z_k$ or $\Delta z_k \Delta x_i$ depending on the direction we solve.

2.2 Implicit centered finite volume scheme for the diffusion equation

The diffusion part consists in the following system:

$$\frac{\partial \rho}{\partial t} = 0\,, \tag{15}$$

$$\frac{\partial (\rho \mathbf{u})}{\partial t} = 0\,, \tag{16}$$

$$\frac{\partial (\rho E)}{\partial t} = -\mathscr{P}_0 \sigma (T^4 - \mathscr{E}_r)\,, \tag{17}$$

$$\frac{\partial \mathscr{E}_r}{\partial t} = \nabla \cdot (\kappa \nabla \mathscr{E}_r) + \sigma (T^4 - \mathscr{E}_r)\,. \tag{18}$$

Since (18) is a heat equation, if we want to use reasonable time step (governed by the Courant Friedrichs Lewy condition (14)), we have to make use of an implicit time discretization.
Writing $E = C_v T + \|\mathbf{u}\|^2/2$, and using (15) and (16), we can show that (17) reduces to an ODE for the temperature T:

$$\rho C_v \frac{\partial T}{\partial t} = -\mathscr{P}_0 \sigma (T^4 - \mathscr{E}_r)\,. \tag{19}$$

The scheme then reads:

$$\rho_i^n C_v \frac{T_i^{n+1} - T_i^n}{\Delta t^n} = -\mathscr{P}_0 \sigma_i^n ((T_i^{n+1})^4 - \mathscr{E}_{r,i}^{n+1})\,, \tag{20}$$

$$\frac{\mathscr{E}_{r,i}^{n+1} - \mathscr{E}_{r,i}^n}{\Delta t^n} - 2\frac{\kappa_{i+1/2}^n \frac{\mathscr{E}_{r,i+1}^{n+1} - \mathscr{E}_{r,i}^{n+1}}{\Delta x_{i+1}} - \kappa_{i-1/2}^n \frac{\mathscr{E}_{r,i}^{n+1} - \mathscr{E}_{r,i-1}^{n+1}}{\Delta x_i}}{(\Delta x_i + \Delta x_{i+1})} = \sigma_i^n ((T_i^{n+1})^4 - \mathscr{E}_{r,i}^{n+1})\,, \tag{21}$$

$$\frac{2}{\kappa_{i+1/2}^n} = \frac{1}{\kappa_i^n} + \frac{1}{\kappa_{i+1}^n}\,.$$

It can be shown by a motonicity argument that this system has a unique solution. It is then solved by the Newton method, mainly because of the nonlinear terms, and the GMRES algorithm ([11]) at each Newton iteration to solve the linear system.

3 Numerical results

In this section, we present numerical simulations of radiative shock solutions. These results are compared to semi-analytic solutions obtained following the method described in [10].
We initialize a Riemann problem setting the left-state (subscript 0) to $\rho_0 = 1, Tr_0 = 1, T_0 = 1, u_0 = \mathscr{M}$, for a given $\mathscr{M}$ (some different values are chosen for the tests), and the right-state (subscript 1) is obtained by solving the so-called "overall jump conditions" ([10]), and taking material and radiative temperatures equal to each other:

$$\rho_0 u_0 = \rho_l u_1 \tag{22}$$

$$\rho_0 u_0^2 + p_0 + \mathscr{P}_0 \frac{T_{r,0}^4}{3} = \rho_1 u_1^2 + p_1 + \mathscr{P}_0 \frac{T_{r,1}^4}{3} \tag{23}$$

$$u_0(\rho_0 E_0 + p_0 + \frac{4}{3}\mathscr{P}_0 T_{r,0}^4) = u_1(\rho_1 E_1 + p_1 + \frac{4}{3}\mathscr{P}_0 T_{r,1}^4) \tag{24}$$

Here, $\kappa = 1$, $\sigma = 10^6$ and $\mathscr{P}_0 = 10^{-4}$.
We take perfect gas equation of state $p = \frac{\rho T}{\gamma}$, with $\gamma = 5/3$.
Figure 1 shows a continuous solution computed over 128 cells. Solutions of figures 2 to 4 undergo discontinuities. For these simulations, a finer mesh is used to capture the solutions.
Numerical and theoretical results are in good agreement. The conservative formulation chosen in (6) seems to be relevant with regard to these particular physical solutions.

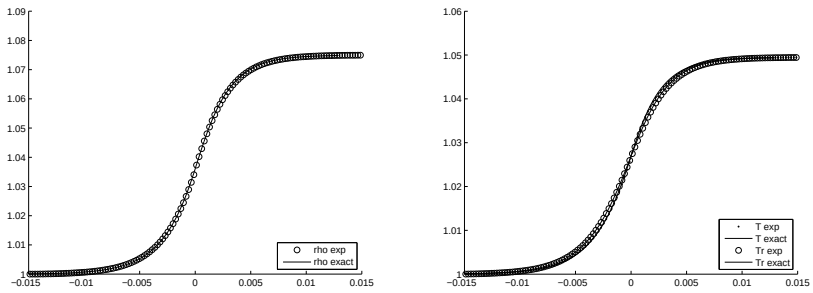

Fig. 1 Solution for density, temperature and radiative temperature for $\mathscr{M} = 1.05$. Comparison with semi-analytic solutions. Number of cells: 128

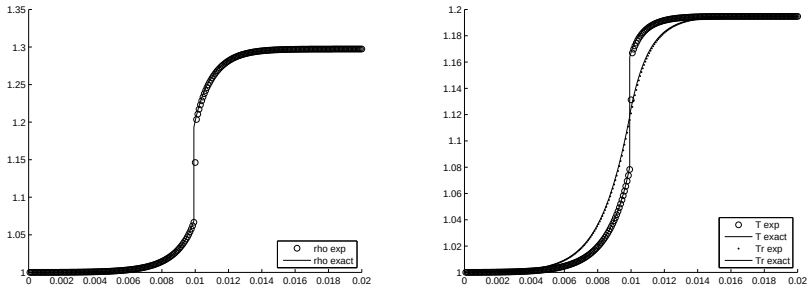

Fig. 2 Solution for density, temperature and radiative temperature for $\mathscr{M} = 1.2$. Comparison with semi-analytic solutions. Number of cells: 256

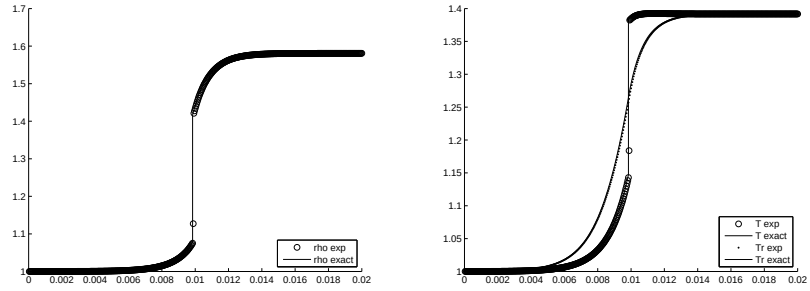

Fig. 3 Solution for density, temperature and radiative temperature for $\mathscr{M} = 1.4$. Comparison with semi-analytic solutions. Number of cells: 512

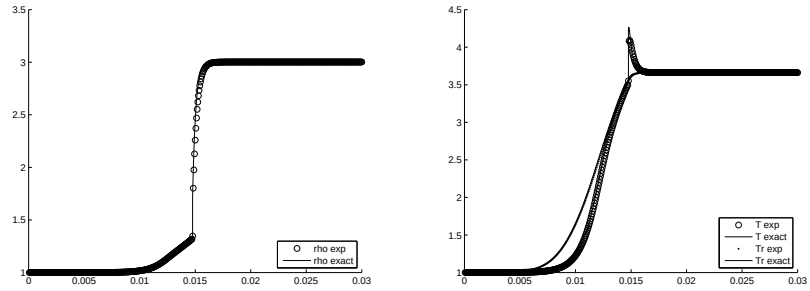

Fig. 4 Solution for density, temperature and radiative temperature for $\mathscr{M} = 3$. Comparison with semi-analytic solutions. Number of cells: 512

4 Conclusion

As said in the introduction, this work is a first step towards the derivation of a multi material $3D$ solver for multi material radiative hydrodynamics. In this paper we have presented our method for the single material case and shown that on physically relevant non trivial solutions, our solver behaves well. The extension for multi material flows is in progress (Chauveheid [3]). The method presented here was designed in order to make this extension as simple as possible. In fact it only remains to extend the so-called condensate techniques of Braeunig *et. al.* [1] to radiative flows.

References

1. Braeunig J.-P., Desjardins B., Ghidaglia J.-M., A totally Eulerian Finite Volume solver for multi-material fluid flows, Eur. J. Mech. B/Fluids, Vol. 28, pp. 475-485, 2009.
2. Buet, C., Despres B.: Asymptotic preserving and positive schemes for radiation hydrodynamic. J. Comput. Phys. **215**, 717–740, 2006.
3. Chauveheid D., Thesis, École normale supérieure, in preparation.
4. Ghidaglia J.-M., Kumbaro A., Le Coq G.: On the numerical solution to two fluid models *via* a cell centered finite volume method, Eur. J. Mech. B/Fluids, 20, 841-867, 2001.
5. Ghidaglia J.-M., Flux schemes for solving monlinear systems of conservation laws, *in* Innovative Methods for Numerical Solution of Partial Differential Equations, Chattot J.J. and Hafez M. Eds, pp 232-242, WORLD SCIENTIFIC, Singapore, 2001.
6. Ghidaglia J.-M., Pascal F.: The normal flux method at the boundary for multidimensional finite volume approximations in CFD. Eur. J. Mech. B/Fluids, Vol. 24(1), pp. 1-17, 2005.
7. Kadioglu, S.Y. , Knoll, D.A., Lowrie, R.B., Rauenzahn, R.M.: A second order self-consistent IMEX method for radiation hydrodynamics. J. Comput. Phys. **229**, 8313–8332, 2010.
8. Kadioglu, S.Y. , Knoll: A fully second order implicit/explicit time integration technique for hydrodynamics plus nonlinear heat conduction problems. J. Comput. Phys. **229**, 3237–3249 (2010)
9. Lowrie, R.B., Morel, J.E.: The coupling of radiation and hydrodynamics. Astrophys. J. **521**, 423–450, 1999.
10. Lowrie, R.B. , Edwards J.D.: Radiative shock solutions with grey nonequilibrium diffusion. Shock Waves **18**, 129–143, 2008.
11. Saad, Y.: Iterative methods for sparse linear systems 2nd edition, SIAM, 2003.

The paper is in final form and no similar paper has been or is being submitted elsewhere.